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1984 J. Phys. A: Math. Gen. 17 3209

(http://iopscience.iop.org/0305-4470/17/16/022)

J. Phys. A: Math. Gen. 17 (1984) 3209-3221. Printed in Great Britain

Anisotropic cubic lattice Potts ferromagnet: renormalisation group treatment

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Received 14 December 1983, in final form 2 April 1984

Abstract. Within a real space renormalisation group framework, we discuss the criticality of the fully anisotropic (arbitrary J_x, J_y and J_z)q-state Potts ferromagnet in the simple cubic lattice. Several previously known exact results for the $d=1$ and $d=2$ particular cases are recovered. Furthermore we obtain: (i) the q-dependence of the $d=3$ correlation length critical exponent ν_3 (in particular, if $q\to 0$, $\nu_3(q)\sim\nu_3(0)+\nu_3'(0)q$ where the present approximate values are $\nu_3(0)\simeq 1.105$ and $\nu_3'(0)\simeq -0.66$; (ii) the q-dependence of the $d=2\leftrightarrow d=3$ crossover critical exponent ϕ_{23} (in particular, $\phi_{23}\propto 1/\sqrt{q}$ if $q\to 0$); (iii) through a convenient numerical extrapolation, a quite accurate proposal for the critical temperatures corresponding to arbitrary ratios J_y/J_x and J_z/J_x and values of q.

1. Introduction

During recent years much work has been devoted to the q-state Potts model, both because of its theoretical richness and its experimental utility (for an excellent review see Wu 1982). However most of this work has been focused on the two-dimensional ($d=2$) case (see Wu 1982, and de Oliveira and Tsallis 1982 and references therein). Some effort has also been devoted to the *isotropic* $d=3$ ferromagnet (Blöte and Swendsen 1979), but we are not aware of any systematic study of the *anisotropic* $d=3$ case and its crossovers to lower dimensions. This is the purpose of the present work (restricted however to the discussion of the critical temperature T_c and correlation length and crossover critical exponents ν and ϕ) which follows along the real space renormalisation group (RG) lines of de Oliveira and Tsallis (1982; which is herein recovered as a particular case). By noting $q_c(d)$ the limiting value of q above which the phase transition is a first-order one (we recall that $\lim_{d\to 1+0} q_c(d)=\infty$, $q_c(2)=4$ and $q_c(3)\leqslant 3$; see Wu (1982) and de Magalhães and Tsallis (1981) and references therein), the present work is restricted to $q\leqslant q_c(d)$. We present in § 2 the model and the formalism, in § 3 the RG results, and in § 4 the extrapolation procedure which provides accurate values for T_c corresponding to models with arbitrary anisotropy.

2. Model and formalism

Let us consider the q-state Potts ferromagnet whose Hamiltonian is given by

$$\mathcal{H} = -q \sum_{(i,j,k)} \left(J_x \delta_{\sigma_{i,j,k}, \sigma_{i+1,j,k}} + J_y \delta_{\sigma_{i,j,k}, \sigma_{i,j+1,k}} + J_z \delta_{\sigma_{i,j,k}, \sigma_{i,j,k+1}} \right) \qquad (J_x > 0;\, J_y, J_z \geqslant 0) \qquad (1)$$

where (i, j, k) runs over all sites of a simple cubic lattice and $\sigma_{i,j,k} = 1, 2, \ldots, q$, $\forall (i, j, k)$. We briefly recall the present status of knowledge of the critically (T_c, ν and ϕ) of this

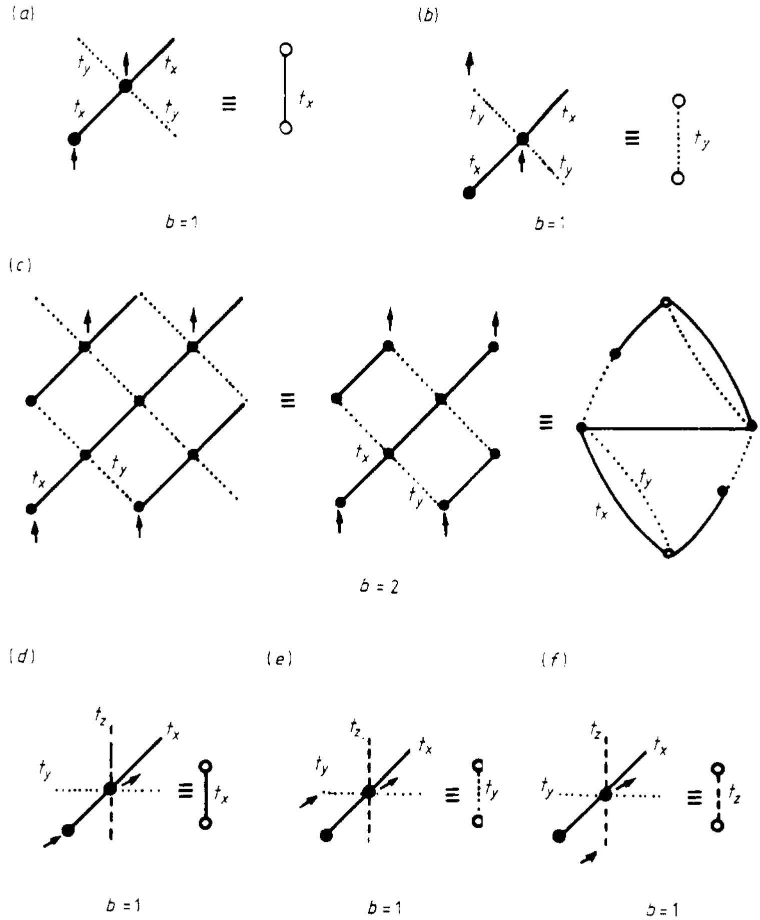

Figure 1. RG cells and their equivalent two-rooted graphs; the arrows indicate the entrance and exit points of the cells; ○ and ● respectively denote terminal and internal nodes of the graphs; t_x, t_y and t_z are the transmissivities along the three crystal-axes. (a), (b) and (c) have been used (de Oliveira and Tsallis 1982) for the $d = 2$ case (the cluster (c) is renormalised into the cluster (a)). (d)–(h) correspond to the $d = 3$ case (the cluster (g), or equivalently the graph (h), is renormalised into the cluster (d)). (g) is the $d = 3$ extension of the central cluster of (c); (h) is the $d = 3$ extension of the right graph of (c); because of its complexity, we have omitted the indication of the $d = 3$ extension of the left cluster of (c).

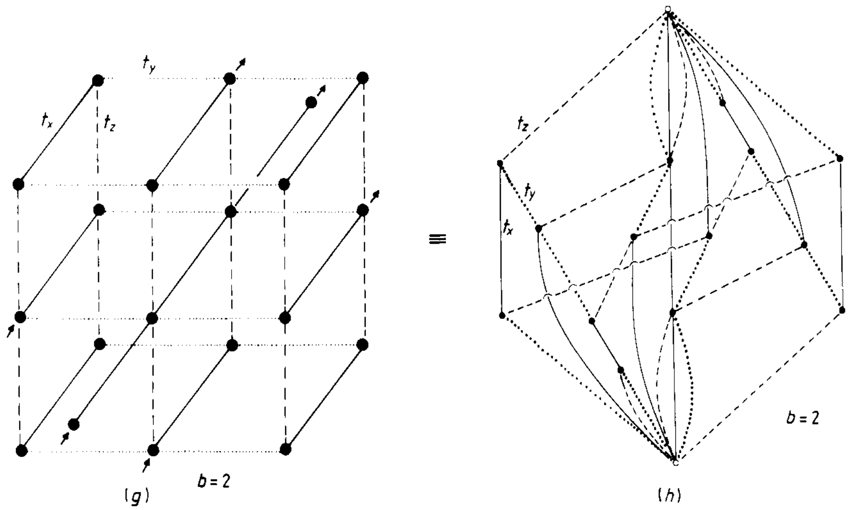

Figure 1. (contd.).

model: (i) for $d=1$ (i.e., $J_y=J_z=0$) the critical temperature T_c vanishes, and the correlation length critical exponent satisfies $\nu_1=1, \forall q$; (ii) for $d=2$ (i.e., $J_z=0$ and $J_y>0$) T_c is exactly known (Baxter *et al* 1978, Burkhardt and Southern 1978 and Hintermann *et al* 1978); (iii) the $d=1$ to $d=2$ and $d=1$ to $d=3$ crossover critical exponents (respectively ϕ_{12} and ϕ_{13}) are commonly believed to satisfy (Redner and Stanley 1979, de Oliveira and Tsallis 1982 and references therein) $\phi_{12}=\phi_{13}=1, \forall q$; (iv) for the isotropic $d=3$ case (i.e., $J_x=J_y=J_z$), T_c is given by

$$k_B T_c/qJ_x \simeq \begin{cases} 3.52\pm 0.05 & \text{for } q=1 \text{ (from Gaunt and Ruskin 1978)} \quad (2a) \\ 2.2556\pm 0.0002 & \text{for } q=2 \text{ (from Zinn-Justin 1979)} \quad (2b) \\ 1.8169 & \text{for } q=3 \text{ (from Jensen and Mouritsen 1979)} \quad (2c) \end{cases}$$

where the $q=1$ value has been obtained from $p_c=0.247\pm 0.003$ by using the Kasteleyn and Fortuin (1969) isomorphism ($p=1-e^{-J_x/k_BT}$) with bond percolation, and where we recall that the $q=3$ case might be slightly first order; the corresponding critical exponent is given by

$$\nu_3 \simeq \begin{cases} 0.88 & \text{for } q=1 \text{ (Heerman and Stauffer 1981)} \quad (3a) \\ 0.630\pm 0.0015 & \text{for } q=2 \text{ (Le Guillou and Zinn-Justin 1980);} \quad (3b) \end{cases}$$

(v) For the $d=2$ to $d=3$ crossover exponent ϕ_{23} the following results are available

$$\phi_{23} \simeq \begin{cases} 1.75 & \text{for } q=1 \text{ (Redner and Stanley 1979)} \quad (4a) \\ 7/4 \text{ (exact)} & \text{for } q=2 \text{ (Liu and Stanley 1972, 1973, Citteur and Kasteleyn 1972, 1973).} \quad (4b) \end{cases}$$

Before presenting our RG formalism, let us define a few convenient variables (Tsallis and Levy 1981, Tsallis 1981):

$$t_\alpha \equiv \frac{1-\exp(-qJ_\alpha/k_BT)}{1+(q-1)\exp(-qJ_\alpha/k_BT)} \in [0, 1] \qquad (\alpha=x, y, z) \qquad (5a)$$

(referred to as *thermal transmissivity*) and

$$s_\alpha^{(d)} \equiv s^{(d)}(t_\alpha) \equiv \frac{\ln[1+(q-1)h(d)t_\alpha]}{\ln[1+(q-1)h(d)]} \in [0, 1] \qquad (\alpha = x, y, z) \tag{5b}$$

where (Tsallis and de Magalhães 1981, de Magalhães and Tsallis 1981) the pure number $h(d)$ sensibly depends on dimensionality d and very slightly on the particular d-dimensional lattice ($h(2) = 1$ for square lattice, and $h(3) = 0.377 \pm 0.044$ for simple cubic lattice).

If we have a series (or parallel) array of two bonds with transmissivities t_1 and t_2, the overall transmissivities (respectively t_s and t_p) are given by $t_s = t_1 t_2$ (series) and $t_p^D = t_1^D t_2^D$ (parallel) where

$$t_i^D \equiv \frac{1-t_i}{1+(q-1)t_i} \qquad (i = 1, 2, \mathrm{p}) \tag{6}$$

(D stands for dual). We can also verify that $h = 1$ (squared lattice) implies $s^{(2)}(t^D) = 1 - s^{(2)}(t)$.

We can now introduce our RG framework. Following along the lines of the de Oliveira and Tsallis (1982) treatment of the square lattice case, we establish the RG recursive relations by renormalising the $b = 2$ cell indicated in figures 1(g), (h) into the $b = 1$ cell in figure 1(d) (b denotes the size of the cell, and coincides with the linear scaling factor). The recurrence is based upon the preservation of the partition function, and can be economically established by using the break-collapse method (Tsallis and Levy 1981). We obtain

$$t_x' = R_b(t_x, t_y, t_z; q) \tag{7}$$

where $R_b(t_x, t_y, t_z; q) = R_b(t_x, t_z, t_y; q)$ is a ratio of polynomials (in the t's) too lengthy to be reproduced herein (the numerator and denominator contain more than 1600 terms each). The sum of the coefficients of the numerator coincides with that corresponding to the denominator and is given (Tsallis and Levy 1981, Essam 1982) by q^κ where $\kappa \equiv$ cyclomatic number $=$ [(number of bonds) $-$ (number of sites) $+ 1$] (for the two-terminal graph of figure 1(h) it is $\kappa = 20$). It is worth noting that $R_b(t_x, t_y, 0; q)$ recovers equation (12) of de Oliveira and Tsallis (1982).

The rest of the RG recursive relations are given by

$$t_y' = R_b(t_y, t_z, t_x; q) \tag{8}$$

$$t_z' = R_b(t_z, t_x, t_y; q) \tag{9}$$

where the equivalence of the x, y and z axes has been taken into account. By studying, for fixed q, the RG flow (in the (t_x, t_y, t_z)-space) determined by equations (7)–(9) we can obtain the fixed points, the para–ferromagnetic separatrix, as well as the relevant Jacobians $\partial(t_x', t_y', t_z')/\partial(t_x, t_y, t_z)$, which in turn determine the critical exponents ν and ϕ.

3. Results

Our results are illustrated in figure 2. Equations (7)–(9) provide the following fixed points: (i) $(s_x^{(2)}, s_y^{(2)}, s_z^{(2)}) = (0, 0, 0)$ and $(1, 1, 1)$ are fully stable, and correspond respectively to the para- and ferromagnetic phases; (ii) $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ are

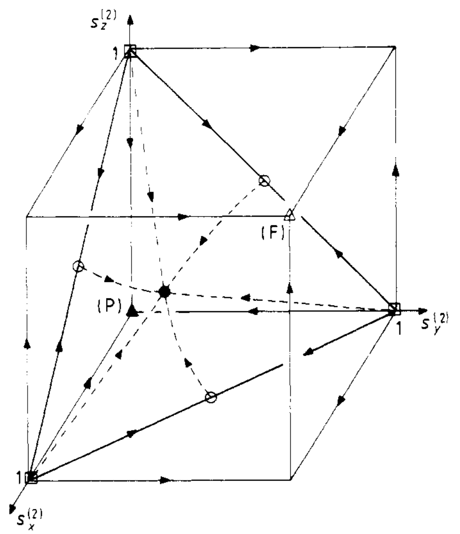

Figure 2. Para(P)-ferro(F) magnetic critical surface in the $(s_x^{(2)}, s_y^{(2)}, s_z^{(2)})$ space. The arrows indicate the RG flow. The main fixed points are indicated: △ (ferromagnetic) and ▲ (paramagnetic) attract all the points respectively above and below the critical surface; □, ○ and ● respectively are the $d=1$, $d=2$ and $d=3$ critical fixed points.

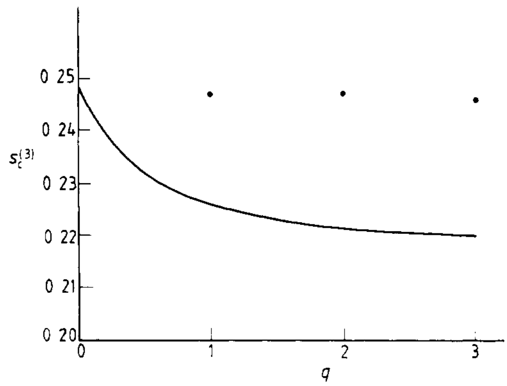

Figure 3. q-dependence of the RG critical point corresponding to the isotropic $d=3$ model (notice the ordinate scale). The dots are series results: $q=1$ (Gaunt and Ruskin 1978), $q=2$ (Zinn-Justin 1979) and $q=3$ (Jensen and Mouritsen 1979).

semi-stable ones, and belong to the ferromagnetic region; (iii) $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are fully unstable ones, and correspond to the $d=1$ case; (iv) $(\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{1}{2}, 0, \frac{1}{2})$ and $(0, \frac{1}{2}, \frac{1}{2})$ are semi-stable ones, and correspond to the $d=2$ isotropic case; (v) $(s_c^{(3)}, s_c^{(3)}, s_c^{(3)})$ is a semi-stable one, and corresponds to the $d=3$ isotropic case ($s_c^{(3)}$ softly depends on q; see figure 3).

The RG critical surface contains the line $s_x^{(2)}+s_y^{(2)}=1$ at $s_z^{(2)}=0$ (and the equivalent ones), thus reproducing the *exact* $d=2$ result. The performance at the isotropic $d=3$ fixed point is not comparable to the $d=2$ case, as the RG provides, for $q=1$, $t_c \simeq 0.2260$ (instead of 0.247, corresponding to equation 2(a)), for $q=2$, $t_c \simeq 0.1949$ (instead of 0.218 11, corresponding to equation 2(b)), and, for $q=3$, $t_c \simeq 0.1750$ (instead of 0.1966, corresonding to equation 2(c)). The results obtained for T_c for arbitrary anisotropy ratios J_y/J_x and J_z/J_x are indicated in table 1.

The Jacobian at the $d=1$ fixed points is fully degenerate and its unique eigenvalue $\lambda^{(1)}$ equals 3. It can be shown that $\lambda^{(1)}=2b-1$ for arbitrary values of b, therefore $\nu_1=\lim_{b\to\infty} \ln b/\ln(2b-1)=1$, thus recovering the *exact* result. The degeneracy of this Jacobian implies that both $d=1 \leftrightarrow d=2$ and $d=1 \leftrightarrow d=3$ crossover exponents ϕ_{12} and ϕ_{13} equal unity, thus recovering the exact answer.

At the $d=2$ fixed points the Jacobians are as follows. Let us analyse for instance the $(s_x^{(2)}, s_y^{(2)}, s_z^{(2)})=(\frac{1}{2}, \frac{1}{2}, 0)$ fixed point (the others are analogous); its Jacobian has the following form

$$\begin{pmatrix} a(q) & b(q) & c(q) \\ b(q) & a(q) & c(q) \\ 0 & 0 & d(q) \end{pmatrix}. \tag{10}$$

Table 1. Critical points ($k_B T_c/qJ_x$) for the anisotropic $d=3$ model: RG (top) and extrapolated (bottom) values. † indicates exact results (see for example Wu 1982) for the isotropic $d=2$ case; ‡, § and ‖ are series results (see the text and figure 3) for the isotropic $d=3$ case.

(*a*) $q=1$

J_z/J_x \ J_y/J_x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0	0.5548	0.7112	0.8349	0.9421	1.0390	1.1287	1.2129	1.2928	1.3692	1.4427
	0	0.5548	0.7112	0.8349	0.9421	1.0390	1.1287	1.2129	1.2928	1.3692	1.4427†
0.1	—	1.0592	1.2643	1.4255	1.5646	1.6896	1.8047	1.9125	2.0145	2.1117	2.2049
	—	1.0229	1.2104	1.3612	1.4934	1.6139	1.7260	1.8317	1.9322	2.0284	2.1210
0.2	—	—	1.4912	1.6689	1.8216	1.9584	2.0840	2.2013	2.3121	2.4174	2.5183
	—	—	1.4061	1.5636	1.7024	1.8293	1.9479	2.0601	2.1670	2.2696	2.3686
0.3	—	—	—	1.8592	2.0223	2.1682	2.3019	2.4266	2.5440	2.6557	2.7625
	—	—	—	1.7259	1.8690	2.0002	2.1231	2.2394	2.3506	2.4575	2.5607
0.4	—	—	—	—	2.1942	2.3476	2.4882	2.6190	2.7421	2.8591	2.9708
	—	—	—	—	2.0158	2.1504	2.2766	2.3963	2.5107	2.6209	2.7273
0.5	—	—	—	—	—	2.5078	2.6543	2.7905	2.9186	3.0402	3.1563
	—	—	—	—	—	2.2882	2.4172	2.5397	2.6569	2.7698	2.8789
0.6	—	—	—	—	—	—	2.8061	2.9472	3.0798	3.2055	3.3256
	—	—	—	—	—	—	2.5489	2.6739	2.7935	2.9088	3.0203
0.7	—	—	—	—	—	—	—	3.0927	3.2294	3.3590	3.4826
	—	—	—	—	—	—	—	2.8012	2.9232	3.0406	3.1543
0.8	—	—	—	—	—	—	—	—	3.3699	3.5031	3.6300
	—	—	—	—	—	—	—	—	3.0472	3.1667	3.2824
0.9	—	—	—	—	—	—	—	—	—	3.6395	3.7696
	—	—	—	—	—	—	—	—	—	3.2882	3.4057
1.0	—	—	—	—	—	—	—	—	—	—	3.9026
	—	—	—	—	—	—	—	—	—	—	3.5250‡

(*b*) $q=2$

J_z/J_x \ J_y/J_x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0	0.4529	0.5708	0.6644	0.7462	0.8205	0.8897	0.9550	1.0172	1.0769	1.1346
	0	0.4529	0.5708	0.6644	0.7462	0.8205	0.8897	0.9550	1.0172	1.0769	1.1346†
0.1	—	0.7276	0.8578	0.9631	1.0555	1.1398	1.2183	1.2925	1.3631	1.4309	1.4964
	—	0.6993	0.8165	0.9141	1.0018	1.0828	1.1591	1.2316	1.3012	1.3682	1.4332
0.2	—	—	0.9967	1.1090	1.2075	1.2972	1.3806	1.4593	1.5341	1.6059	1.6751
	—	—	0.9323	1.0299	1.1184	1.2009	1.2792	1.3540	1.4261	1.4958	1.5634
0.3	—	—	—	1.2270	1.3304	1.4243	1.5116	1.5939	1.6721	1.7470	1.8192
	—	—	—	1.1275	1.2164	1.2998	1.3791	1.4553	1.5289	1.6002	1.6696
0.4	—	—	—	—	1.4379	1.5355	1.6262	1.7115	1.7926	1.8702	1.9449
	—	—	—	—	1.3058	1.3898	1.4700	1.5472	1.6219	1.6944	1.7652
0.5	—	—	—	—	—	1.6365	1.7301	1.8181	1.9018	1.9818	2.0588
	—	—	—	—	—	1.4746	1.5556	1.6336	1.7092	1.7828	1.8546

Table 1—continued

(*b*) $q=2$

J_z/J_x \ J_y/J_x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.6	—	—	—	—	—	—	1.8264	1.9169	2.0029	2.0851	2.1642
	—	—	—	—	—	—	1.6373	1.7162	1.7927	1.8671	1.9399
0.7	—	—	—	—	—	—	—	2.0098	2.0979	2.1821	2.2630
	—	—	—	—	—	—	—	1.7958	1.8732	1.9485	2.0221
0.8	—	—	—	—	—	—	—	—	2.1880	2.2741	2.3568
	—	—	—	—	—	—	—	—	1.9513	2.0274	2.1019
0.9	—	—	—	—	—	—	—	—	—	2.3619	2.4463
	—	—	—	—	—	—	—	—	—	2.1044	2.1796
1.0	—	—	—	—	—	—	—	—	—	—	2.5323
	—	—	—	—	—	—	—	—	—	—	2.2556§

(*c*) $q=3$

J_z/J_x \ J_y/J_x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0	0.4060	0.5066	0.5869	0.6572	0.7213	0.7813	0.8380	0.8923	0.9445	0.9950
	0	0.4060	0.5066	0.5869	0.6572	0.7213	0.7813	0.8380	0.8923	0.9445	0.9950†
0.1	—	0.6044	0.7072	0.7918	0.8670	0.9360	1.0008	1.0622	1.1211	1.1778	1.2327
	—	0.5820	0.6749	0.7537	0.8251	0.8917	1.9548	1.0150	1.0730	1.1291	1.1836
0.2	—	—	0.8144	0.9028	0.9814	1.0537	1.1214	1.1857	1.2472	1.3065	1.3638
	—	—	0.7642	0.8413	0.9123	0.9791	1.0429	1.1043	1.1636	1.2213	1.2774
0.3	—	—	—	0.9945	1.0761	1.1510	1.2212	1.2878	1.3515	1.4128	1.4721
	—	—	—	0.9174	0.9878	1.0547	1.1187	1.1807	1.2408	1.2993	1.3565
0.4	—	—	—	—	1.1602	1.2374	1.3098	1.3783	1.4439	1.5070	1.5681
	—	—	—	—	1.0580	1.1249	1.1892	1.2515	1.3122	1.3714	1.4293
0.5	—	—	—	—	—	1.3168	1.3911	1.4614	1.5287	1.5934	1.6560
	—	—	—	—	—	1.1918	1.2564	1.3191	1.3802	1.4399	1.4985
0.6	—	—	—	—	—	—	1.4671	1.5391	1.6080	1.6742	1.7382
	—	—	—	—	—	—	1.3213	1.3843	1.4459	1.5061	1.5652
0.7	—	—	—	—	—	—	—	1.6127	1.6830	1.7506	1.8159
	—	—	—	—	—	—	—	1.4478	1.5098	1.5705	1.6301
0.8	—	—	—	—	—	—	—	—	1.7546	1.8235	1.8900
	—	—	—	—	—	—	—	—	1.5723	1.6335	1.6936
0.9	—	—	—	—	—	—	—	—	—	1.8936	1.9613
	—	—	—	—	—	—	—	—	—	1.6952	1.7557
1.0	—	—	—	—	—	—	—	—	—	—	2.0300
	—	—	—	—	—	—	—	—	—	—	1.8169‖

The eigenvalues $\lambda_1^{(2)} = a(q) + b(q)$ and $\lambda_2^{(2)} = a(q) - b(q)$ recover the results of de Oliveira and Tsallis (1982) and $\lambda_3^{(2)} = d(q)$ (too lengthy to be reproduced herein; it monotonically decreases from about 8.1 to about 3.3 when q increases from 0 to 3). The respective eigenvectors are $(1, 1, 0)$ $(1, -1, 0)$ and $(1, 1, (\lambda_3^{(2)}(q) - \lambda_1^{(2)}(q))/c(q))$. We verify that $\lambda_1^{(2)}(q) \geqslant 1 \geqslant \lambda_2^{(2)}(q) > 0$, $\forall q \geqslant 0$, and that $\lambda_3^{(2)}(q) \geqslant \lambda_1^{(2)}(q) (\lambda_3^{(2)}(q) < \lambda_1^{(2)}(q))$ if $q \leqslant q^* (q > q^*)$ where $q^* \simeq 5$. The coefficient $c(q)$ monotonically increases from roughly zero to roughly 10 when q varies from zero to infinity; consistently the eigenvector associated with $\lambda_3^{(2)}(q)$ is roughly along the $(1, 1, 1)$ direction for q varying let us say between 1 and 3. Within the present $b = 2$ RG approximation the critical exponents are given by $\nu_2 = \ln 2/\ln \lambda_1^{(2)}$ and $\phi_{23} = \ln \lambda_3^{(2)}/\ln \lambda_1^{(2)}$: see figures 4 and 5 and table 2.

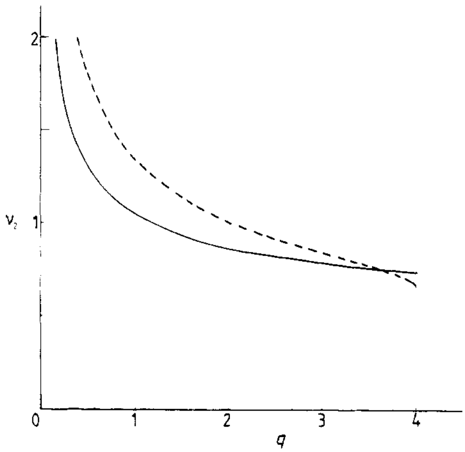

Figure 4. q-dependence of the $d = 2$ correlation length critical exponent ν_2: RG (——) and exact (- - -; den Nijs 1979).

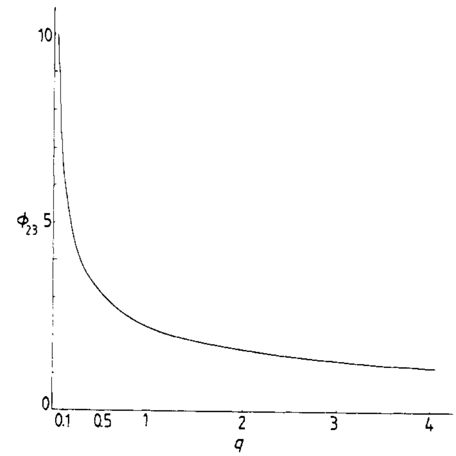

Figure 5. q-dependence of the $d = 2 \leftrightarrow d = 3$ RG crossover exponent ϕ_{23}. The dots are series (●; Redner and Stanley 1979) and exact (○; Liu and Stanley 1979, 1973, Citteur and Kasteleyn 1972, 1973).

The Jacobian at the $d = 3$ fixed point $(t_x = t_y = t_z = t_c^{(3)}(q))$ is as follows:

$$\begin{pmatrix} e(q) & f(q) & f(q) \\ f(q) & e(q) & f(q) \\ f(q) & f(q) & e(q) \end{pmatrix}. \tag{11}$$

The eigenvalues are

$$\lambda_1^{(3)} = e(q) + 2f(q) \tag{12}$$

and

$$\lambda_2^{(3)} = \lambda_3^{(3)} = e(q) - f(q) \tag{13}$$

and the eigenvectors are respectively $(1, 1, 1)$ and any vector perpendicular to $(1, 1, 1)$. We verify $\lambda_1^{(3)}(q) \geqslant 1 \geqslant \lambda_2^{(3)}(q) > 0$, $\forall q \geqslant 0$. The corresponding approximated critical exponent is given by $\nu_3 = \ln 2/\ln \lambda_1^{(3)}$ (see figure 6 and table 2); $\lambda_2^{(3)}(q)$ monotonically increases from roughly zero to 1 when q varies from zero to infinity.

Table 2. Present RG and exact (or series) results for the critical point t_c and exponents ν and ϕ for the isotropic d-dimensional models. [a] Wu (1982) and references therein; [b] den Nijs (1979); [c] Redner and Stanley (1979); [d] Liu and Stanley (1972, 1973), Citteur and Kasteleyn (1972, 1973); [e] Gaunt and Ruskin (1978); [f] Zinn-Justin (1979); [g] Jensen and Mouritsen (1979); [h] Heerman and Stauffer (1981); [i] Le Guillou and Zinn-Justin (1980).

			$q\to 0$	$q=1$	$q=2$	$q=3$	$q=4$
$d=1$	$t_c^{(1)}$	RG ($\forall b$)	1	1	1	1	1
		exact	1[a]	1[a]	1[a]	1[a]	1[a]
	ν_1	RG ($\forall b$)	$\frac{\ln b}{\ln(2b-1)}$	$\frac{\ln b}{\ln(2b-1)}$	$\frac{\ln b}{\ln(2b-1)}$	$\frac{\ln b}{\ln(2b-1)}$	$\frac{\ln b}{\ln(2b-1)}$
		exact	1[a]	1[a]	1[a]	1[a]	1[a]
	ϕ_{1d}	RG ($\forall b$)	1	1	1	1	1
		exact	1[a]	1[a]	1[a]	1[a]	1[a]
$d=2$	$t_c^{(2)}$	RG ($\forall b$)	$\sim 1-\sqrt{q}$	1/2	$\sqrt{2}-1$	$1/(\sqrt{3}+1)$	1/3
		exact	$\sim 1-\sqrt{q}$[a]	1/2[a]	$\sqrt{2}-1$[a]	$1/(\sqrt{3}+1)$[a]	1/3[a]
	ν_2	RG ($b=2$)	$\frac{45\ln 2}{52\sqrt{q}}\simeq\frac{0.600}{\sqrt{q}}$	1.042	0.864	0.785	0.738
		exact	$\frac{\pi}{3\sqrt{q}}\simeq\frac{1.047}{\sqrt{q}}$[b]	$\frac{4}{3}\simeq 1.333$[b]	1[b]	$\frac{5}{6}\simeq 0.833$[b]	$\frac{2}{3}\simeq 0.667$[b]
	ϕ_{23}	RG ($b=2$)	$\simeq\frac{2}{\sqrt{q}}$	2.258	1.637	1.346	1.163
		exact or series	—	1.75[c]	1.75[d]	—	—
$d=3$	$t_c^{(3)}$	RG ($b=2$)	$\simeq 0.294-0.11q$	0.2260	0.1949	0.1750	—
		series	—	0.247[e]	0.21811[f]	0.1966[g]	—
	ν_3	RG ($b=2$)	$\simeq 1.105-0.66q$	0.756	0.657	0.606	—
		series	-	0.88[h]	0.630[i]	—	—

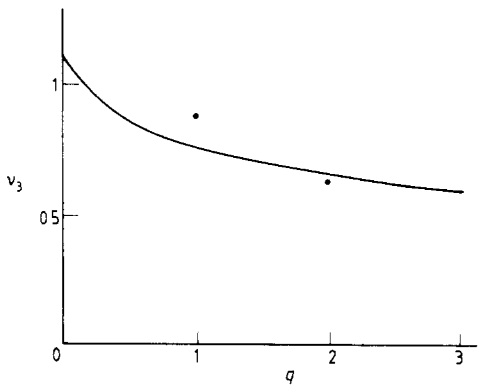

Figure 6. q-dependence of the $d=3$ correlation length critical exponent ν_3: RG (——) and series (●; Heerman and Stauffer 1981 for $q=1$; Le Guillou and Zinn-Justin 1980 for $q=2$).

4. Extrapolation for the critical point

In this section we describe an *ad hoc* extrapolation procedure for the critical temperature T_c for an arbitrary value of q. We take advantage from the fact that the anisotropic $d=2$ RG result is the *exact* one for all q, and that the isotropic $d=3$ RG result is not too bad (at least for $q=1,2,3$, where comparison with other results is possible). It essentially consists in 'pushing' the centre ($s_x^{(2)}=s_y^{(2)}=s_z^{(2)}=s_c^{(2)}$) of the RG critical surface in the ($s_x^{(2)}$, $s_y^{(2)}$, $s_z^{(2)}$)-space (see figure 2), until it coincides (by imposition) with the best value (noted s_0; usually from series) available in the literature for that particular value of q; the effects of this 'pushing' monotonically and softly decrease while going from the centre of the critical surface to its periphery, eventually vanishing on the anisotropic $d=2$ limiting case (i.e. $s_x^{(2)}=0$ or $s_y^{(2)}=0$ or $s_z^{(2)}=0$) where, as said before, the exact result is reproduced by the RG. As no confusion can occur in the present section, we use $s_\alpha \equiv s_\alpha^{(2)}$ ($\alpha = x, y, z$), where $s_\alpha^{(2)}$ is given by equation (5b) with $h(2)=1$. Summarising, the input, for a given q, of the extrapolation procedure is the RG critical surface *and* the 'exact' value for the isotropic $d=3$ critical point.

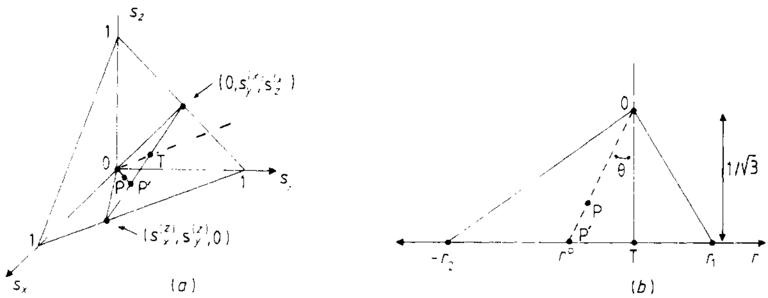

Figure 7. Geometric constructions related to the extrapolation procedure (see § 4): (*a*) the ($s_x^{(2)}$, $s_y^{(2)}$, $s_z^{(2)}$) space; (*b*) the triangle determined by the points O, P and T of (*a*).

We consider, in the (s_x, s_y, s_z)-space (see figure 7(*a*)), the point P (on the RG critical surface and not belonging to the trisectrix $s_x=s_y=s_z$) to be extrapolated; its coordinates are noted (s_x^P, s_y^P, s_z^P) and conventionally satisfy $1 \geq s_x^P \geq s_y^P \geq s_z^P \geq 0$ (every other region is directly associated with this one through trivial symmetry transformations). This point and the trisectrix determine a unique plane whose equation is given by

$$\frac{s_y - s_z}{s_x - s_z} = \frac{s_y^P - s_z^P}{s_x^P - s_z^P} \equiv g \in [0, 1]. \tag{14}$$

This plane and the plane

$$s_x + s_y + s_z = 1 \tag{15}$$

(which contains all three *exact* $d=2$ critical lines, e.g., $s_x+s_y=1$ for $s_z=0$) determine a unique straight line. This line cuts the $s_z=0$ plane at the point ($s_x^{(z)}$, $s_y^{(z)}$, 0) and the $s_x=0$ plane at the point (0, $s_y^{(x)}$, $s_z^{(x)}$), where

$$s_x^{(z)} = 1/(1+g), \qquad s_y^{(z)} = g/(1+g). \tag{16a, b}$$

$$s_y^{(x)} = (1-g)/(2-g), \qquad s_z^{(x)} = 1/(2-g). \tag{16c, d}$$

This line also cuts the trisectrix at the point T with coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. If we consider now the triangle determined by the points $(0, 0, 0)$, $(s_x^{(z)}, s_y^{(z)}, 0)$ and $(0, s_y^{(x)}, s_z^{(x)})$ (see figure 7(*b*)), we immediately obtain

$$r_1 = +[(\tfrac{1}{3})^2 + (\tfrac{1}{3} - s_y^{(x)})^2 + (\tfrac{1}{3} - s_z^{(x)})^2]^{1/2} \tag{17a}$$

$$r_2 = +[(\tfrac{1}{3} - s_x^{(z)})^2 + (\tfrac{1}{3} - s_y^{(z)})^2 + (\tfrac{1}{3})^2]^{1/2} \tag{17b}$$

where r_1 and r_2 are defined in figure 7(*b*) (r_1 and r_2 respectively correspond to $(s_x^{(z)}, s_y^{(z)}, 0)$ and $(0, s_y^{(x)}, s_z^{(x)})$). The angle θ defined in figure 7(*b*) is determined by

$$\cos\theta = \frac{s_x^{\mathrm{P}} + s_y^{\mathrm{P}} + s_z^{\mathrm{P}}}{\sqrt{3}\, s^{\mathrm{P}}} \tag{18}$$

where

$$s^{\mathrm{P}} \equiv +[(s_x^{\mathrm{P}})^2 + (s_y^{\mathrm{P}})^2 + (s_z^{\mathrm{P}})^2]^{1/2}. \tag{19}$$

The quantity r^{P} defined in figure 7(*b*) is given by

$$r^{\mathrm{P}} = -(\tan\theta)/\sqrt{3} \in [-r_2, r_1]. \tag{20}$$

Obviously $s^{\mathrm{P}} \leqslant [\frac{1}{3} + (r^{\mathrm{P}})^2]^{1/2}$.

The value s^{P} is going to be extrapolated into s^{ex} through the relation

$$s^{\mathrm{ex}} = s^{\mathrm{P}}[1 + F(r^{\mathrm{P}})] \tag{21}$$

where the extrapolating function $F(r)$ is assumed to satisfy the following conditions:

(i) $\quad F(r_1) = F(-r_2) = 0 \qquad (22a)$

(ii) $\quad F(0) = \sqrt{3}s_0/s^{\mathrm{P}} - 1, \qquad (22b)$

(iii) $\quad F(r)$ maximal at $r = 0$. $\qquad (22c)$

The simplest polynomial which satisfies these conditions is

$$F(r) = F(0)(1 - Ar^2 - Br^3) \tag{23}$$

where

$$A \equiv (r_2^3 + r_1^3)/(r_1^2 r_2^3 + r_1^3 r_2^2) \tag{24a}$$

and

$$B \equiv (1 - Ar_1^2)/r_1^3. \tag{24b}$$

Finally the coordinates of the extrapolated point are given by

$$s_\alpha^{\mathrm{ex}} = (s^{\mathrm{ex}}/s^{\mathrm{P}})s_\alpha^{\mathrm{P}} \qquad (\alpha = x, y, z). \tag{25}$$

In spite of its apparent complexity, the implementation in computer of this extrapolating algorithm is very simple. The operational steps are as follows: (i) given $(s_x^{\mathrm{P}}, s_y^{\mathrm{P}}, s_z^{\mathrm{P}})$, g is calculated through equation (14), and also $s_x^{(z)}$, $s_y^{(z)}$, $s_y^{(x)}$ and $s_z^{(x)}$ through equations (16), hence r_1 and r_2 (through equations (17)) and finally A and B (through equations (24)); (ii) $(s_x^{\mathrm{P}}, s_y^{\mathrm{P}}, s_z^{\mathrm{P}})$ also determine θ and s^{P} through equations (18) and (19), which in turn determine r^{P} through equation (20); (iii) s_0 (taken from the literature) and s^{P} determine $F(0)$ through equation (22*b*); (iv) the knowledge of A, B, r^{P} and $F(0)$ determines $F(r^{\mathrm{P}})$ through equation (23), hence s^{ex} (through equation (21)) and finally $(s_x^{\mathrm{ex}}, s_y^{\mathrm{ex}}, s_z^{\mathrm{ex}})$ through equation (25).

The results obtained by using the above algorithm are indicated in table 1. In order to test the reliability of our results we have compared them with series calculations available for $q=1$ (figure 8(a)) and $q=2$ (figure 8(b)) for the particular cases $0 \leq J_z/J_x \leq J_y/J_x = 1$ and $0 \leq J_y/J_x = J_z/J_x \leq 1$. The agreement is very satisfactory (the discrepancy in the t-variable is always smaller than 0.01).

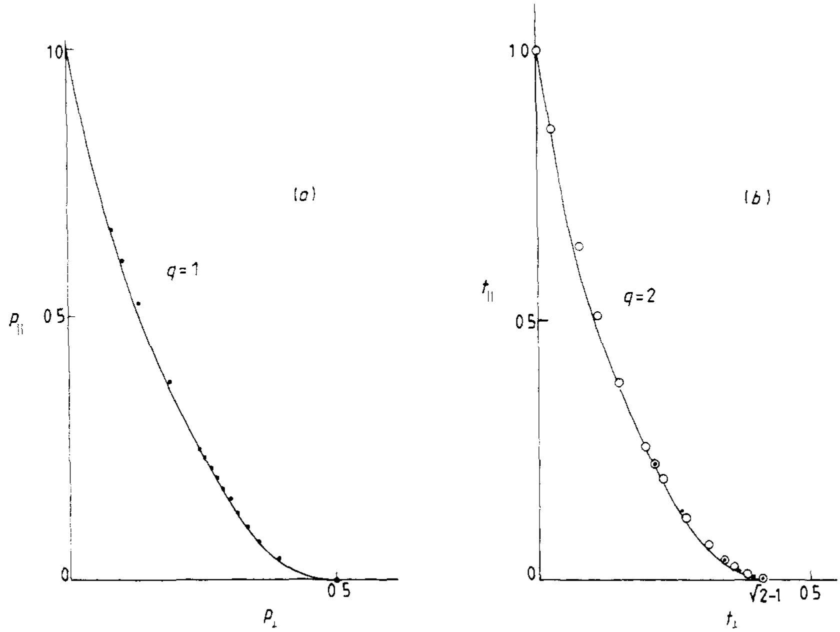

Figure 8. Present extrapolated results (——) for the critical point corresponding to the particular anisotropic $d=3$ case where two coupling constants are assumed equal ($\equiv J_{\perp}$) and the third one ($\equiv J_{\parallel}$) eventually different. We have the isotropic $d=1$, $d=2$ and $d=3$ cases at the ordinate, abcissa and bisectrix respectively. (a) $q=1$; the dots are series results (Redner and Stanley 1979); (b) $q=2$; both dots (Oitmaa and Enting 1971) and circles (Paul and Stanley 1972) are series results.

5. Conclusion

We have discussed, within a real space renormalisation group framework, the q-state Potts ferromagnet in the fully anisotropic (arbitrary J_x, J_y and J_z) simple cubic lattice. The q-dependences of the critical temperature T_c, the one-, two- and three-dimensional correlation length critical exponents ν_1, ν_2 and ν_3, and the $d=1 \leftrightarrow d>1$ and $d=2 \leftrightarrow d=3$ crossover critical exponents ϕ_{1d} and ϕ_{23} are analysed in the second-order phase transition region ($\forall q$ for $d=1$, $q \leq 4$ for $d=2$, and $q \leq q_c(3) \simeq 3$ for $d=3$).

The present renormalisation group reproduces a considerable amount of already known *exact* results such as $t_c^{(1)} = \nu_1 = \phi_{1d} = 1$, $\forall q$, for $d=1$, $t_c = 1/(\sqrt{q}+1)$ for $d=2$, etc; it also recovers, in the $q \to 0$ limit, the correct asymptotic behaviour $\nu_2 \propto 1/\sqrt{q}$. Whenever our numerical results do not coincide with available exact or series ones, the discrepancies are acceptable. Furthermore the universality classes we obtain are as commonly expected, i.e. the $d=3$ one for all values of J_x, J_y and J_z as long as none of them vanishes, and the $d=2$ one when only one among them vanishes. The general picture inspires reasonable confidence, and therefore we tend to believe that the $q \to 0$

$d=3$ results $\phi_{23} \propto 1/\sqrt{q}$, $t_c^{(3)}(q) \sim t_c^{(3)}(0) + t_c^{(3)\prime}(0)q$ and $\nu_3(q) \sim \nu_3(0) + \nu_3'(0)q$ (with finite values for $t_c^{(3)}(0)$, $t_c^{(3)\prime}(0)$, $\nu_3(0)$ and $\nu_3'(0)$) are correct.

We have also developed an extrapolation procedure for T_c which has proved to be quite satisfactory whenever comparison with other available results (typically from series) was possible, namely for the $0 \leq J_z/J_x \leq J_y/J_x = 1$ and $0 \leq J_y/J_x = J_z/J_x \leq 1$ particular cases of the $q = 1, 2$ models. Through this procedure we have calculated T_c for arbitrary ratios J_y/J_x and J_z/J_x and values of q (the $q = 3$ results are probably almost unaffected by the fact that the transition might be slightly first order). A theory which, enlarging the parameter space, would succeed in recovering the existence of first-order phase transitions would be very welcome. If alternatively the present RG is understood as referring to the hierarchical lattice defined by figure 1(h), then all the results it provides are exact for $q \geq 0$.

Acknowledgments

Useful remarks from A C N de Magalhães, E M F Curado, M Schick, P M C de Oliveira, P R Hauser and W K Theumann are gratefully acknowledged. One of us (LRS) has benefited from a CNPq Fellowship (Brazilian Agency); CT acknowledges partial support through the tenure of a Guggenheim Fellowship.

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